Throughout his professional career Frege was focused on proving the truths of arithmetic from simpler logical principles. But Frege’s motivation for this project remains obscure. Generally we can understand the motivation behind a project, and its importance, by understanding the implications it was thought to have by its creator. So we must ask ourselves, what would a successful reduction of arithmetic to logic imply? And what consequences does the apparent impossibility of consistently constructing this reduction have? By answering these questions we will uncover Frege’s motivations. Of course I don’t want to imply that Frege had only a single motivation for his project, given the effort he spent on it I suspect that he had a number of reasons for pursuing it. What I am after then is a kind of primary motivation, one that can explain the entire project and which is consistent with the assumptions behind it. And, most importantly, it must be one that gives the project at least the possibility for success, such that if a reduction of arithmetic to logic was possible, and if Frege’s account had included that version of the reduction, then it would have achieved what it set out to do.

A number of authors have already addressed this problem, among them Benacerraf, Jeshion, Kitcher, and Weiner, and their solutions fall into basically two camps. Kitcher and Weiner understand Frege’s project as being motivated primarily by epistemological concerns and thus as having implications for our understanding of how we know, or might know, mathematical truths, while Benacerraf and Jeshion describe Frege’s project as being motivated primarily by mathematical concerns and thus as not really having philosophical consequences,
as having them only by a kind of happy accident. Since it is natural to understand Frege as a philosopher let us begin by examining how Frege might be understood to be motivated by epistemological, and thus philosophical, concerns.

1: An Epistemological Motivation

Obviously understanding the motivation behind Frege’s project isn’t trivial, and the problems in doing so are aggravated by the tendency of certain empiricists, who used Frege’s work for their own ends, to project their epistemological motivations onto him, as Benacerraf points out in his paper “Frege: The Last Logicist”. The empiricists saw synthetic a priori truths as contradictory to their project, because their existence would seem to imply that there are ways of arriving at knowledge about things, knowledge not contained in the ideas we use to describe them, without any experiences of them. Thus, if the empiricist project was to succeed, these philosophers needed to explain all apparent cases of synthetic a priori truths as either analytic or a posteriori. Given this aim the reduction of arithmetic to logic seems eminently desirable, because it will show that the truths of arithmetic are really analytic truths, and, together with the reduction of the rest of mathematics to arithmetic, this will vindicate their project by clearing up these apparent epistemological difficulties. But Frege certainly did not see his own work in this light. Although Frege originally thought that arithmetic was analytic he had no problem with synthetic a priori truths in general, and he refers to geometry as an example of such truths on a number of occasions. So, whatever Frege’s motivation was, it was certainly not to vindicate some kind of empiricist project.
But just because Frege wasn’t part of an empiricist project doesn’t mean that the motivations behind his project weren’t epistemological. Let us turn then to Kitcher’s explanation of Frege’s motivation as detailed in his paper “Frege’s Epistemology”. Specifically Kitcher understands his project as a search for a way to better justify the truths of arithmetic. Of course Kitcher wouldn’t claim that this is all Frege’s project does, his systematization of the subject was probably also meant to provide a certain clarity and certainty that was missing in other treatments of arithmetic. However, to explain why arithmetic was to be reduced to logic we must understand Frege as building on an existing epistemological foundation, namely that of Kant. Whether something is analytic or synthetic has significant implications in Kant’s epistemology, because the way we justify such claims, and how certain we can be about them, differs between the two, and reducing arithmetic to logic would make it synthetic and thus improve its epistemological status. What Frege can be understood as attempting to do then is to correct a small problem with Kant’s project, which classified arithmetical truths as synthetic a priori but gave them an uncertain basis in pure intuition, thus making these truths seem less certain than Frege thought they should be.

If that is indeed what Frege is doing then he must see some problem with the idea that arithmetic is synthetic a priori, and indeed Frege makes a number of remarks seemingly to this effect in Foundations, where he implies that there is something wrong with our intuitions about numbers, that somehow they are less clear than our geometrical intuitions (for example, in Foundations section 13). Unfortunately, despite this supporting evidence, there is a problem with interpreting Frege in this way. Suppose that his goal really was to make what we take to be arithmetical truths more certain. How might a reduction of arithmetic to logic accomplish this? One possibility is that it might reveal that certain statements that are taken to be arithmetical
theorems are inconsistent either with each other or with the arithmetical axioms. But obviously that could be revealed by any formal and precise treatment of the arithmetical axioms, and thus can’t be the motivation behind a reduction of arithmetic to logic. So if the reduction is supposed to do anything for us it must be because there is a possibility of revising the axioms themselves, and that if we can conduct the reduction to logic without modifying those axioms their current form is thus justified. Unfortunately that too seems impossible, since if Frege had come up with definitions that led to some other arithmetical axioms then we would be inclined to deny that he had really captured numbers with his definitions. And indeed Frege himself states, in Foundations section 70, that he will demonstrate the correctness of his definitions by showing that they validate the usual arithmetical axioms, which seems to rule out the possibility of proving the correctness of the axioms by means of these definitions without circularity. Nor would the ability to construct this reduction make us more certain that arithmetical axioms were free from inconsistency, since, as Frege himself discovered, logic itself may turn out to be inconsistent. Thus I would invoke the principle mentioned in the introduction, and argue that we should abandon this understanding of Frege’s motivation, given that even a successful reduction of arithmetic to logic can’t improve the certainty of the arithmetical axioms.

But this isn’t the only way to understand Frege as having a primarily epistemological motivation. Weiner, in her paper “The Philosopher Behind The Last Logicist”, also develops an account of Frege’s motivation along those lines. However, Weiner does not see Frege as attempting to improve our epistemological position with respect to arithmetical truths, rather she understands him as trying to understand it, to explain how it is, exactly, that we come to know facts about numbers. Frege then is after the sources of our knowledge, at least when it comes to arithmetic. This then is why the analytic/synthetic distinction is emphasized by Frege in
Foundations, according to Weiner, not because he is trying to build on Kant, but because Frege has defined these terms via reference to the grounds of proofs, and thus to establish whether arithmetic is analytic or synthetic determines what axioms it is proved from, and thus where knowledge of it ultimately springs from.

However, this interpretation of Frege’s project runs into problems when we consider how Frege gets form his logical axioms to the theorems of arithmetic; he must introduce definitions of the number zero, what it means to be a number in general, the succession of numbers, and so on. The problem is that these definitions are not necessitated by his logical foundation, indeed they can’t be because they contain terms that are not included in that foundation. Therefore it would seem that any theorems of arithmetic yielded by this system are grounded not only on the logical laws, but on the correctness of these definitions as well, introducing and additional and unexplained source of knowledge that would seem to frustrate this project. But maybe this only seems like a problem, because it might be argued that Frege only needs to establish the possibility of a logical proof of the arithmetical axioms. He states in the preface to Begriffsschrift that the grounds of a proposition are determined by the most perfect method of proof for it, and, while the knowledge that what Frege has shown is a proof of the arithmetical axioms rests on the definitions, the proof itself does not. Unfortunately this avoids only part of the problem, because, as Frege admits in that preface, there is nothing inconsistent with a proposition having a purely logical proof and yet that we come to apprehend it only through the operation of our senses. What this means is that, while Frege is certainly trying to establishing something interesting about the nature of the arithmetical axioms by showing that a logical proof might be constructed for them, something which I will explore in section three, how we come to know them is a separate matter. How we come to know them, and thus the sources of our
knowledge, may be tied up with how we come to apprehend them and our psychological faculties\(^1\). Thus constructing a successful reduction of arithmetic in this way would not reveal how we can know arithmetical truths, but rather why they are true, which is something rather different.

But that is not the entirety of Weiner’s proposal, and she partly avoids this problem by suggesting that Frege’s definitions are not actually justified by our understanding of what numbers are. Indeed they are not epistemologically justified at all. What Frege is attempting to do, according to Weiner, is not to repair or improve our existing conception of arithmetic, but instead replace it with a new systematic science, which can yield theorems analogous to those of arithmetic, and thus which can effectively replace it. In this new systematic science the definitions of zero, number, and so on are not justified, but are rather simply stipulative abbreviations for more complicated constructions, which happen to prove fruitful by yielding a number of interesting formulas. But, while this avoids the dilemma described above, it raises new problems for understanding Frege’s motivation as epistemic, because it undercuts that very motivation. If Frege’s motivation was to understand how we can know the truths of arithmetic, or how we do know them, he will have failed under this account, because what he has done is created a new system, arithmetic\((F)\), which replaces our less rigorous arithmetic\((O)\), and thus he has explained only how we know the truths of arithmetic\((F)\), but not those of arithmetic\((O)\). And it was how we know the truths of arithmetic\((O)\) that was the interesting question, because when we create a new system, essentially from scratch, there is no question about how we come to

\(^1\) Let me put this point another way, if Frege’s motivation was epistemological then he should be either trying to improve our epistemological practices with respect to arithmetic, so that we become more certain of it axioms, or he should be trying to explain how the practices we actually use to arrive at these theorems yield knowledge, if we do know them. Weiner seems to be saying that he is attempting to do the second but, as has just been established, that is clearly not the case because if we consider the definitions as something extra then the proofs Frege has provided clearly don’t reflect how we know arithmetical theorems, but at best say why they are the case, which is not an epistemological result.
know things about it. Granted we might later come to believe that the two systems are identical, or that arithmetic(O) is in some way a subset of arithmetic(F), but at best that would inform us why arithmetic(O) was in fact the case, not how we knew it to be the case. Again, this is surely something interesting about arithmetic(O), but it doesn’t shed light on how we knew it to be the case before Frege came along, and is thus incompatible with understanding Frege as seeking to discover the sources of that knowledge.

2: A Mathematical Motivation

Since the way Frege actually proceeds makes ascribing an epistemological motivation to him difficult perhaps we should stop trying to understand Frege’s project as a philosophical one and instead understand it as first and foremost a mathematical project. Indeed this is what Benacerraf suggests in “Frege: The Last Logicist”, mentioned previously; that Frege’s project was primarily a mathematical one, and thus had primarily mathematical motivations. Specifically, he claims that Frege’s motivation stems from concerns with the foundations of arithmetic, and mathematics in general. Frege comments, in the introduction to Foundations, that mathematics is currently pursued in a haphazard fashion, with premises accepted simply because they have yet to be shown to give rise to any contradictions. He describes this as a merely empirical certainty, and implies that it should be obvious to us that mathematics should not be done in this way. This, it is claimed, explains Frege’s motivation behind the introduction of his begriffsschrift (his logical notation), which will allow mathematicians to produce completely rigorous and gap-free proofs. And the reduction of arithmetic to logic, according to
Benacerraf, is thus simply a manifestation of his mathematician’s curiosity concerning the relationships of truths.

And that is not the only way of understanding Frege as being driven primarily by a mathematical motivation. In “Frege’s Notions of Self-Evidence” Jeshion claims that Frege’s primary motivation is to prove the theorems of arithmetic, which are themselves not self-evident (a special status accorded to certain propositions), from axioms that are. In doing so we become more confident of the theorems justified in this way, and thus Frege’s project also conveys epistemological benefits as a side-effect, even though its primary role is as part of a “Euclidean” mathematical project. This might seem to make Jeshion’s interpretation fall prey to some of the same objections that faced Kitcher’s, that Frege’s project seems to presuppose the truth of certain facts about arithmetic, and Weiner’s, that in getting to the truths of arithmetic from logic we are forced to employ definitions that are not themselves logical truths. But Jeshion’s interpretation has ways of avoiding these problems. First, Jeshion does not understand Frege as taking claims that are self-evident to be beyond any possible shadow of doubt. Thus Frege’s project is not meant to establish the truths of arithmetic with a perfect certainty where before they might be doubted. Instead a proof from self-evident truths is only intended to improve our epistemological and mathematical position. The idea, I suppose, is that by producing such a proof of a proposition that was thought to be true we increase our certainty, because of the certainty we have in each of the self-evident links in that chain. However, no self-evident proposition is immune from being overturned, and things that were thought to be self-evident may later turn out to be in need of proof or analysis themselves. But this still leaves us with the problematic definitions, which are key components in this chain. Jeshion has an explanation for this as well, she claims that the definitions too may have the property of being self-evident, and
indeed any proper definition, in which the senses of the terms on both sides is the same, is. Thus Frege’s defense of his definitions is explained, not as an attempt to prove them, but as an attempt to explain them so that their self-evidence can become apparent.

I would not deny the correctness of the accounts of Frege’s motivation given by Benacerraf and Jeshion, at least for the most part. Clearly Frege’s project was motivated by certain mathematical problems and general curiosity. However, while a mathematical motivation can explain some of Frege’s work it does not suffice to explain everything, especially when it comes to understanding why arithmetic was to be reduced to logic. First of all, from a purely mathematical standpoint, there doesn’t seem to be much of a difference between the axioms of arithmetic and the axioms of geometry, and yet Frege never attempted to reduce geometry to logic, or even suggested that the geometrical axioms needed proving. This strongly suggests that Frege’s arithmetical target was chosen at least in part because of philosophical concerns. Secondly Foundations seems entirely superfluous if Frege was motivated only mathematically. The work of actually proving arithmetic from logic is taken up in Basic Laws, and if that was Frege’s sole concern why write Foundations at all? Furthermore, Foundations contains a large number of refutations of other definitions of numbers. If Frege was motivated primarily by a desire to prove further facts from his logical system, possibly to extend chains of self-evidence, there would be no need for such arguments; certainly few other mathematicians, setting out to explore the consequences of some axioms, first argue that all other ways at arriving at the claims they seek to prove are illegitimate in some way. The fact that Frege does spend this extra effort implies that his project is more than strictly mathematical, and that these other definitions of number threaten to undermine it. Finally, if Frege’s motivations were entirely mathematical, why wasn’t he satisfied to prove the truths of arithmetic from Hume’s principle, or
from some axiomatization of arithmetic, which is certainly sufficiently self-evident and precise for all mathematical purposes? Again, this strongly suggests that Frege’s desire to reduce arithmetic to logic, and not just to some simpler principles, was a result of more than a mathematical motivation.

3: An Ontological Motivation

So, while understanding Frege’s project as epistemologically significant hasn’t panned out, neither has understanding it as purely mathematically motivated. If we were to let matters rest with this it would seem to imply that Frege’s reduction of arithmetic to logic was pointless given Frege’s other positions and the way he actually conducted that reduction, and thus that in a certain pragmatic sense we might simply ignore it when considering Frege’s work since it apparently has no significance except as a subject that prompts Frege to say interesting things about logic and language in other contexts. But there is something strange in the idea that the primary task Frege put himself to was a fool’s errand given his insight into other matters. Thus I am prompted to put forward a third way of understanding the significance of Frege’s project, one that attributes to Frege what could be called an ontological motivation.

As I understand Frege he had two motivations for his project, not one, that often overlapped. The problem is, for modern readers, that one of Frege’s motivations is something we simply no longer care so much about, and so we overlook its importance for Frege. Specifically, I claim that Frege was partly motivated to vindicate his begriffsschrift, and to argue for the mathematical value of rigorous and gap free proofs, even when they don’t seem absolutely necessary. Obviously this isn’t a motivation we can share, because we are fully
convinced of the value of such systems when there is some question as to the validity of the proof, and we are equally aware that when we want to actually get somewhere in mathematics we must put them somewhat to one side because they often make proofs a bit too complex for mere mortals to handle. But this is not something that Frege’s contemporaries understood. Frege had introduced his system and believed that using it would improve mathematics. But obviously no mathematician was simply going to pick it up and start doing mathematics with it without some reason to believe that they would benefit from it.

So, from Frege’s point of view, what was needed was to establish that the kinds of detailed proofs that could be proved by the begriffsschrift were desirable, as well as a demonstration of the value of the begriffsschrift. Thus proving the truths of arithmetic was useful in this mathematical enterprise, because, while they had not been satisfactorily proved before from other principles, neither was there any question about their validity. Understanding Frege’s motives in this way explains why he places such a heavy emphasis on the desirability of precise mathematical proofs in general even when they may not seem needed, because if we desire such proofs, and Frege can provide them with his begriffsschrift, it demonstrates to us the mathematical value of that system, even if we didn’t need it to validate those truths or to justify the certainty we attribute to them. We might understand this as a pragmatic motivation, but it is also a mathematical motivation, and explains why Frege claimed over and over again that mathematical proofs must proceed only by absolutely primitive and self-evident steps, even if proceeding in that way won’t have any obvious epistemological benefits. But, as a mathematical motivation it leaves many of the same questions open as the mathematical motivations discussed previously, namely why he went after arithmetic instead of geometry, what the purpose of
Foundations was, and why he wasn’t satisfied with proofs grounded in Hume’s principle or some other simple axiomatization.

To answer those questions we must appeal to Frege’s other motivation: to understand what numbers are, which I describe as ontological. To some this might appear an absurd way to interpret Frege, but certainly Frege’s project can certainly be described as a reduction of arithmetic to logic, and it isn’t unreasonable to argue that, in general, when we reduce one thing to another it entails ontological consequences, namely that what is reduced turns out to really be essentially the same kind of thing as that it was reduced to. Naturally this doesn’t mean that Frege’s motivation was necessarily ontological, it could be simply a side-effect of his project.

But he begins his introduction to Foundations by considering answers that might be given to the question “what is a number?” and why they are unsatisfactory. And, similarly, in the appendix to Basic Laws volume 2 he states that what justifies us in taking the numbers as objects is a primary problem of arithmetic, and to solve that problem seems, at the very least, to require us to know what kind of objects numbers are. Together such remarks indicate that ontological questions were at least one of the problems Frege was directing his attention towards.

But, on the other hand, Frege does not make such remarks often, although that doesn’t necessarily mean that they didn’t constitute his fundamental motivation. Thus the best possible evidence we might be able to find to support the claim that Frege had an ontological motivation, given a lack of textual evidence, will be if taking Frege to have such a motivation explains the way he actually conducts his project without leaving questions unanswered, as taking him to have a mathematical motivation did. For example, if we grant Frege an ontological motivation then the reason that much of Foundations is devoted to arguing against alternate definitions of number is obvious; Frege wants to know what numbers are and finds these other definitions
unsatisfactory. Moreover, if someone accepts them it will clearly prevent them from coming to an adequate understanding of what numbers are, and thus for Frege’s ontological goals to be accomplished he must first show that what numbers are hasn’t been satisfactorily addressed in some other way. It also explains why Frege was not satisfied with Hume’s principle as a foundation for arithmetic. The problem is, essentially, that while it says when numbers are equal it does not say what numbers are, it does not define them. And since Frege is after such definitions it will not satisfy him, even if he could have proved all the truths of arithmetic from it. Finally this ontological motivation also explains, in conjunction with some of his comments about geometry, why he was not similarly interested in proving those truths. Frege thought that the objects of geometry are given to us in our pure intuitions of space, and thus that, as they are given to us, we already know what the objects of geometry are. Thus there are no significant ontological questions to answer of the kind that might provide primitive definitions to construct proofs of the axioms from. And so we can see that taking Frege to have an ontological motivation in conjunction with a mathematical motivation neatly answers the questions taking him to have a mathematical motivation raised.

However, the fact that taking Frege to have an ontological motivation answers those questions is not sufficient by itself. Taking Frege to have an epistemological motivation also would have answered those questions, but earlier I argued that the way Frege conducts his project makes it unsuitable to answering those questions. Thus to successfully argue that Frege had an ontological motivation it must be demonstrated Frege’s methods could provide answers to ontological questions. Whether it can is not immediately obvious, as some might argue that Frege’s definitions are essentially stipulative or are fundamentally unjustified, and thus cannot be used to defend a position concerning what kind of objects numbers are. Let us consider then
what a challenge to Frege on this point would entail; a claim that he has not arrived at the nature of numbers, or not sufficiently justified his assertions, and thus has not developed a genuine solution to any ontological problems. Certainly it couldn’t be claimed that numbers are anything but logical entities, assuming we are convinced by his earlier arguments, which Frege takes to demonstrate that the arithmetical truths must be analytic (or at least give us good reasons for believing them to be). Because, under Frege’s definition of analyticity, to be analytic is simply to be proven from logical truths and while such proofs can draw out new consequences they cannot introduce new objects. Thus if arithmetical truths are analytic numbers must be logical objects. Nor could it be claimed that numbers are indefinable, and thus necessarily of an unknown nature, simply because Frege has provided at least a possible candidate for such a definition. But now it might seem that the ontological questions have been answered too quickly, because the actual reduction of arithmetic to logic was not needed to get to that point, and, given that Frege himself admits that other definitions of numbers might be given in logical terms and used to prove the arithmetical truths, this might seem to make that part of his project unmotivated by ontological concerns. However, those facts alone don’t imply that his definitions and subsequent proofs of the arithmetical axioms are useless in an ontological enterprise. Rather they serve to confirm the argument that arithmetic is analytic, which was previously given in a rather weak way, namely by arguing against the other possibilities. By actually proving arithmetical truths from logical principles we would thus confirm the analyticity of arithmetic, and thus the ontological claims, beyond any possible shadow of doubt. Indeed the fact that this is where Frege’s project ran into problems validates his methods as good ones for answering ontological questions concerning numbers, because given that the reduction is unsuccessful we might conclude that numbers aren’t logical objects. And this strongly indicates
that Frege’s methods genuinely lead us to answers to our ontological questions about numbers, because they can’t easily be manipulated to provide whatever answer is desired.

Thus understanding Frege to have an ontological motivation has virtues that taking him to have an epistemological or mathematical one lack. Frege’s project simply can’t answer epistemological questions, at least not given the way Frege proceeds, and it does too much to be purely a mathematical project. But taking him to have an ontological motivation fits perfectly with his project; it explains it so that we can understand the project as neither arbitrary nor as containing superfluous parts. More importantly his project is able to answer ontological questions about numbers; even its failure reveals that number are not logical objects, an ontological fact, and thus that they must be something else. Of course the weakness of this understanding of Frege’s motivation is that, while he makes many remarks that can be taken to imply that he has an epistemological or purely mathematical motivation, there are only a few places where Frege can be read as admitting to having an ontological motivation. But I submit that this may have been simply because he thought that it was obvious what the value of a reduction of arithmetic to logic was. Just as I have neglected to provide this paper with a long preamble about the importance of understanding Frege’s motivation perhaps he also felt that the investigation itself was what was really important, and that its consequences would simply reveal themselves naturally. Given the long history of empiricists who took Frege’s work in a completely different sense than it was intended I guess he was mistaken.

References


